Select Solutions (Triple integrals)

(2d) Compute $\iiint_R (x-y)\,dV$ where R is the region bounded by

$$z = x^2 - 1, \quad z = 1 - x^2, \quad y = 0, \quad \text{and} \quad y = 2.$$

Sol: Parametrizing the region R:

Pro-Tip: Try to draw a picture of the region.

Note $x^2 - 1 = 1 - x^2 \iff 2x^2 = 2 \iff x = \pm 1$

Also: sign chart $+$ | -1 | $-$ | 1 | $+$: $(x^2-1)-(1-x^2) = 2x^2 - 2$ yields $x^2 - 1 \leq 1 - x^2$ on the region.

$$\therefore \quad R = \{(x,y,z) : 0 \leq y \leq 2,\ -1 \leq x \leq 1,\ x^2 - 1 \leq z \leq 1 - x^2\}$$

Computing the Integral:

$$\therefore \iiint_R (x-y)\,dV = \int_{y=0}^{2}\int_{x=-1}^{1}\int_{z=x^2-1}^{1-x^2} (x-y)\,dz\,dx\,dy$$

$$= \int_{y=0}^{2}\int_{x=-1}^{1} (x-y)\, z\Big|_{z=x^2-1}^{1-x^2}\,dx\,dy$$

$$= \int_{y=0}^{2}\int_{x=-1}^{1} (x-y)\left((1-x^2)-(x^2-1)\right)dx\,dy$$

$$= \int_{y=0}^{2}\int_{x=-1}^{1} (x-y)(2-2x^2)\,dx\,dy$$

$$= \int_{y=0}^{2}\int_{x=-1}^{1} \left(2x - 2x^3 - 2y + 2x^2 y\right)dx\,dy$$

$$= \int_{y=0}^{2} \left[x^2 - \tfrac{1}{2}x^4 - 2xy + \tfrac{2}{3}x^3 y\right]_{x=-1}^{1} dy$$

$$= \int_{y=0}^{2} \left(\left(1 - \tfrac{1}{2} - 2y + \tfrac{2}{3}y\right) - \left(1 - \tfrac{1}{2} + 2y - \tfrac{2}{3}y\right)\right) dy$$

$$= \int_{y=0}^{2} -\tfrac{16}{3}y\,dy = -\tfrac{8}{3}y^2\Big|_{y=0}^{2} = -\tfrac{8}{3}(9-0) = -24 \quad \blacksquare$$

(29) Compute $\iiint_R x\,dV$ for R bounded by $x = 4y^2 + 4z^2$ and $x = 4$.

Sol: The given surfaces have intersection:

$$4 = 4y^2 + 4z^2 \iff y^2 + z^2 = 1.$$

Thus we make a change to cylindrical coords.

Pro-tip: Make a sketch of what's going on...

Substitution: $\begin{cases} y = r\cos(\theta) \\ z = r\sin(\theta) \\ x = x \end{cases}$ $\therefore\ dV_{xyz} = r\,dV_{r\theta x}$,

$R_{r,\theta x} = \{(r,\theta,x) : 0 \le r \le 1, 0 \le \theta \le 2\pi, 4r \le x \le 4\}$

Computation:
$$\iiint_R x\,dV = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{x=4r}^{4} x r\,dx\,d\theta\,dr$$
$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} \tfrac{1}{2} r x^2\Big|_{x=4r}^{4}\,d\theta\,dr$$
$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} \tfrac{1}{2} r(16 - 16r^2)\,d\theta\,dr$$
$$= \int_{r=0}^{1} 8r(1-r^2)\ \theta\Big|_{\theta=0}^{2\pi}\,dr$$
$$= \int_{r=0}^{1} 8r(1-r^2)(2\pi - 0)\,dr \qquad \left(\begin{array}{ll} u = 1-r^2 & u(1) = 0 \\ du = -2r\,dr & u(0) = 1 \end{array} \right.$$
$$= \int_{u=1}^{0} -8\pi u\,du$$
$$= -4\pi u^2\Big|_{u=1}^{0} = -4\pi(0-1) = 4\pi$$

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(4d) Compute $\iiint_R (x-y)\,dV$ for R the region between cylinders $x^2+y^2=1$ and $x^2+y^2=16$ and above $z=0$ and below $z=y+4$.

Sol: We make a switch to cylindrical coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases} \quad \therefore\ dV_{xyz} = r\,dV_{r\theta z}$$

Pro-tip: Draw a picture!

$$R_{xyz} = \{(x,y,z) : 1 \le x^2+y^2 \le 16,\ 0 \le z \le y+4\}$$

$$\therefore R_{r\theta z} = \{(x,y,z) : 1 \le r \le 4,\ 0 \le \theta \le 2\pi,\ 0 \le z \le r\sin(\theta)+4\}$$

Finally we compute:

$$\iiint_R (x-y)\,dV = \int_{r=1}^{4}\int_{\theta=0}^{2\pi}\int_{z=0}^{r\sin(\theta)+4} (r\cos(\theta) - r\sin(\theta))\,r\ dz\,d\theta\,dr$$

$$= \int_{r=1}^{4}\int_{\theta=0}^{2\pi} r^2(\cos(\theta)-\sin(\theta))\ z\Big|_{z=0}^{r\sin(\theta)+4} d\theta\,dr$$

$$= \int_{r=1}^{4}\int_{\theta=0}^{2\pi} r^2(\cos(\theta)-\sin(\theta))(r\sin(\theta)+4)\,d\theta\,dr$$

$$= \int_{r=1}^{4}\int_{\theta=0}^{2\pi} r^2\left(r\cos(\theta)\sin(\theta) + 4\cos(\theta) - r\sin^2(\theta) - 4\sin(\theta)\right) d\theta\,dr$$

$$= \int_{r=1}^{4}\int_{\theta=0}^{2\pi} r^2\left(\frac{r}{2}\sin(2\theta) + 4\cos(\theta) - \frac{r}{2}(1-\cos(2\theta)) - 4\sin(\theta)\right) d\theta\,dr$$

$$= \int_{r=1}^{4} r^2\left(-\frac{r}{4}\cos(2\theta) + 4\sin(\theta) - \frac{r}{2}\left(1-\frac{1}{2}\sin(2\theta)\right) + 4\cos(\theta)\right)\Big|_{\theta=0}^{2\pi} dr$$

$$= \int_{r=1}^{4} r^2\left(\left(-\frac{r}{4} + 0 - \frac{r}{2}(1-0) + 4\right) - \left(-\frac{r}{4} + 0 - \frac{r}{2}(1-0) + 4\right)\right) dr$$

$$= \int_{r=1}^{4} 0\,dr = 0$$

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(6d) Compute the volume of the solid between $z = 24 - x^2 - y^2$ and $z = \sqrt{x^2+y^2}$

Sol: Let's use cylindrical coordinates:

Pro-tip: Draw a picture!

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases} \quad \therefore \; dV_{xyz} = r\, dV_{r\theta z}$$

with this substitution, we have $r \le z \le 24 - r^2$.

Now the curve of intersection is:

$$r = 24 - r^2 \iff r^2 + r - 24 = 0 \iff r = \frac{-1 \pm \sqrt{1 + 4\cdot 24}}{2}$$

Reject negative root (it would imply z - negative).

$\therefore$ the upper bound on r is $\alpha = \dfrac{\sqrt{97} - 1}{2}$.

Hence we integrate:

$$\mathrm{Vol}(R) = \iiint_R 1\, dV = \int_{r=0}^{\alpha} \int_{\theta=0}^{2\pi} \int_{z=r}^{4-r^2} dz\, d\theta\, dr$$

$$= \int_{r=0}^{\alpha} \int_{\theta=0}^{2\pi} (4 - r^2 - r)\, d\theta\, dr$$

$$= \int_{r=0}^{\alpha} 2\pi (4 - r - r^2)\, dr$$

$$= 2\pi \left(4r - \tfrac{1}{2} r^2 - \tfrac{1}{3} r^3\right)\Big|_{r=0}^{\alpha}$$

$$= 2\pi \left(4\alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{3}\alpha^3\right) \quad \leftarrow \text{that's good enough.}$$

$$= 2\pi\left(2(\sqrt{97} - 1) - \tfrac{1}{8}(\sqrt{97} - 1)^2 - \tfrac{1}{24}(\sqrt{97} - 1)^3\right)$$

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(6f) Compute the volume of the solid lying above the cone $\varphi = \frac{\pi}{3}$ and inside the sphere $\rho = 4\cos(\varphi)$.

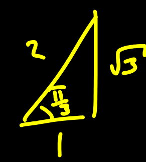
Sol: Let's use spherical coordinates (note that's how they're describing the region). Pro-tip: Picture !! ☺

$$R = \{(\rho, \theta, \varphi) : 0 \le \rho \le 4\cos(\varphi),\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{3}\}$$

$dV_{xyz} = \rho^2 \sin(\varphi)\, dV_{\rho\theta\varphi}$

Hence $\text{Vol}(R) = \iiint_R 1\, dV = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\frac{\pi}{3}} \int_{\rho=0}^{4\cos(\varphi)} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\frac{\pi}{3}} \frac{1}{3}\sin(\varphi)\, \rho^3 \Big|_{\rho=0}^{4\cos(\varphi)} d\rho\, d\varphi\, d\theta$$



$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/3} \frac{1}{3}\sin(\varphi)\left(64\cos^3(\varphi)\right) d\varphi\, d\theta$$

$$\begin{cases} u = \cos(\varphi) & u(\frac{\pi}{3}) = \frac{1}{2} \\ du = -\sin(\varphi)\, d\varphi & u(0) = 1 \end{cases}$$

$$= \int_{\theta=0}^{2\pi} \int_{u=1}^{1/2} -\frac{64}{3} u^3\, du\, d\theta$$

$$= \int_{\theta=0}^{2\pi} -\frac{16}{3} u^4 \Big|_{u=1}^{1/2} d\theta$$

$$= \int_{\theta=0}^{2\pi} -\frac{16}{3}\left(\frac{1}{16} - 1\right) d\theta$$

$$= 5\int_{\theta=0}^{2\pi} d\theta = 5 \cdot 2\pi = 10\pi$$

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